Algebraic Topology Qual

May 21, 2018

Problem 1. Suppose X is a path-connected space with universal covering space X'. Prove that if X' is compact then $\pi_1(X)$ is finite.

Problem 2. Find a Δ-complex structure for the Klein bottle and compute its simplicial homology with coefficients in $\mathbb{Z}$.

Problem 3.

- What is $H_i(S^3; \mathbb{Q})$ for $i \geq 0$? Just the answer; no justification necessary.
- A closed 3-manifold M is called a *rational homology* 3-*sphere* if $H_i(M; \mathbb{Q}) \cong H_i(S^3; \mathbb{Q})$ for all i. Prove (using a combination of Poincaré duality and the Universal Coefficient Theorem) that a closed 3-manifold M is a rational homology 3-sphere iff $H_1(M; \mathbb{Q}) = 0$.

Problem 4. Let $X = S^1 \vee S^1 \vee S^1$ be the wedge of three circles shown below. Let x, y, z be the three loops indicated in the figure. Let $W = X \cup_{f_1} e_1^2 \cup_{f_2} e_2^2$ be the space obtained from X by attaching one 2-cell via the map

$$f_1 : \partial e_1^2 \to X$$

which sends the boundary to the loop $xyx^{-1}zy^{-1}$; and attaching another 2-cell via the map

$$f_2 : \partial e_2^2 \to X$$

which sends the boundary to the loop z^7.

- Describe the associated cellular chain complex for W (including the boundary maps).
- Compute $H^i(W; \mathbb{Z}/2\mathbb{Z})$ for all $i \geq 0$.

Topology Qual, Differential Geometry:

Summer 2018

Please show all your work. You may use any results proved in class or on HW.

(1) Let X be the vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = x^2 + y^2$.
  (a) Compute $df(X)$ in terms of the standard coordinates x, y on $\mathbb{R}^2$. (Please begin by computing df.)
  (b) Compute $X(f)$ in terms of x, y and verify that you get the same answer as in part (a).

(2) (a) If α is a differential form on a manifold M, then must it be true that $\alpha \wedge \alpha = 0$? Prove or provide a counterexample.
  (b) If α and β are closed differential forms, prove that $\alpha \wedge \beta$ is closed.
  (c) If, in addition (i.e., continue to assume α is closed), β is exact, prove that $\alpha \wedge \beta$ is exact.

(3) Recall that $SL(2, \mathbb{R})$ is the set of 2×2 real matrices with $\det = 1$. Prove that we can realize $SL(2, \mathbb{R})$ as an imbedded submanifold of $\mathbb{R}^4$ of dimension 3.

(4) Let V and W be smooth vector fields on a smooth manifold M.
  (a) Briefly explain what is meant by $VW : C^\infty(M) \to C^\infty(M)$. That is, given $f \in C^\infty(M)$, how does one produce $VW(f) \in C^\infty(M)$?
  (b) Recall that a map $X : C^\infty(M) \to \mathbb{R}$ is said to be a *derivation* at $p \in M$ if it is $\mathbb{R}$–linear and satisfies

$$X(fg) = f(p)Xg + g(p)Xf$$

  for all $f, g \in C^\infty(M)$.
  For $p \in M$, let $\mathrm{ev}_p : C^\infty(M) \to \mathbb{R}$ be the map

$$\mathrm{ev}_p(f) := f(p).$$

  Is

$$(\mathrm{ev}_p) \circ (VW) : C^\infty(M) \to \mathbb{R}$$

  necessarily a derivation for each $p \in M$? Prove or give a counterexample.
  (c) Briefly explain what is meant by $[V, W] : C^\infty(M) \to C^\infty(M)$.
  (d) Is

$$(\mathrm{ev}_p) \circ [V, W] : C^\infty(M) \to \mathbb{R}$$

  necessarily a derivation for all $p \in M$? Prove or give a counterexample.