ANALYSIS QUALIFYING EXAM

JUNE. 2014

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Exercise 1. (30 points.)
Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ measurable. Show that

$$\int_X |f(x)|^p d\mu(x) = \int_0^\infty p t^{p-1} \varphi(t) dt,$$

where $\varphi(t) = \mu\{x | t < |f(x)|\}$.

Exercise 2. (30 points.)

(1) Prove that not every subset of $[0,1]$ is Lebesgue measurable
(2) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of Lebesgue measurable functions. Prove that the set $E = \{x | \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is Lebesgue measurable

Exercise 3. (30 points.)
Let X, Y be Banach spaces. If $T : X \to Y$ is a linear map such that $f \circ T \in X^*$ for all $f \in Y^*$ then T is bounded.

Exercise 4. (30 points.)
Let $(X, \mathcal{M}, \mu)$ be a finite measure space. For each of the following claims prove or give a counter example:

(1) If a sequence (f_n) of real valued measurable functions on X converges μ a.e., then (f_n) converges in measure.
(2) If a sequence (f_n) of real valued measurable functions on X converges in measure., then (f_n) converges μ a.e.
(3) If a sequence (f_n) of real valued measurable functions on X is Cauchy in $L^1(\mu)$, then (f_n) converges in measure.