Analysis Qualifying Exam ——Spring 2011

Please answer all 6 problems from each section and show your work. Each problem is worth 30 points.

SECTION 1: REAL ANALYSIS

In your proofs, you may use any major theorem, except the fact you are trying to prove, or a mere variant of it. State clearly what theorems you use. Good luck.

1 Let $f_n : X \to \mathbb{R}$ be (X, μ) measurable functions such that $0 \leq f_n \leq 1$ and μ is a finite measure. Prove that if

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \mu(X)$$

then $f_n \to 1$ a.e.

2 Let f be Lebesgue integrable on $(0, a)$ and $g(x) = \int_x^a t^{-1} f(t)dt$.
Prove that g is Lebesgue integrable on $(0, a)$ and $\int_0^a g(x)dx = \int_0^a f(x)dx$.

3 Let μ, ν be finite measure on (X, M) with $\nu << \mu$. Define $\lambda = \mu + \nu$ and $f = \frac{d\nu}{d\lambda}$.
Prove that $0 \leq f < 1$ μ−a.e. and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

4 Let E be a Borel set in $\mathbb{R}^n$ and m denote Lebesgue measure on $\mathbb{R}^n$. Let

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B(r, x))}{m(B(r, x))}$$

whenever the limit exists. Prove that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

5 Let X, Y be Banach spaces and $T : X \to Y$ a linear map such that $f \circ T \in X^*$ for all $f \in Y^*$. Prove that T is bounded.

6
a) Prove the Minkowski inequality $||f + g||_p \leq ||f||_p + ||g||_p$ for $1 \leq p < \infty$.
b) Show that $L^p(X)$ is complete for $1 \leq p < \infty$.

SECTION 2: COMPLEX ANALYSIS

In your proofs, you may use any theorem from the syllabus for Complex Analysis, except of course you may not use the fact you are trying to prove, or a mere variant of it. State clearly what theorems you use. Good luck.

$\mathbb{E}$ is the open unit disc. $\mathbb{N}$ is the set of positive integers.

1 TRUE or FALSE. If TRUE, prove it; if FALSE, give a counterexample. Let f be an analytic function on $\mathbb{E} - \{a\}$.

a) f has a removable singularity at a implies that e^f has a removable singularity at a.

b) f has a pole at a implies that e^f has a pole at a.

c) f has an essential singularity at a implies that e^f has an essential singularity at a.

2 Let $u : \mathbb{C} \to \mathbb{R}$ be harmonic. You may assume u is C^∞.

a) Show there exists $v : \mathbb{C} \to \mathbb{R}$ harmonic such that $u + iv$ is entire.

b) Suppose u is bounded. Must u be constant? Proof or counterexample.

3 Let $f : \mathbb{C} \to \mathbb{C}$ be analytic, nonconstant and doubly periodic with periods ω_1, ω_2.

a) Prove that $\omega_2/\omega_1 \notin \mathbb{R}$.

b) Let P_a be a period domain such that no zero or pole of f lies on its boundary ∂P_a. Interpret

$$\frac{1}{2\pi i} \int_{\partial P_a} \frac{z f'(z)\, dz}{f(z)}$$

in terms of the zeros and poles of f and prove that it is an element of $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.

4 Let $s \in \mathbb{C}$ and consider

$$\sum_{n=1}^{\infty} n^{-s}.$$

a) Prove that the sum converges absolutely and uniformly in the set $\{s \mid \Re(s) \geq \delta\}$ for any $\delta > 1$.

b) Assume the result in part a). Then the sum defines an analytic function on $U := \{s \mid \Re(s) > 1\}$ which we will call $\zeta(s)$. Prove that for $s \in U$,

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx.$$

Be sure to justify any interchange of infinite sum and integral you perform.

Hint: use an integral formula for $\Gamma(s)$.

c) Using the formula in b) and Cauchy's theorem, it is not hard to prove that

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz$$

where $(-z)^{s-1}$ is defined on the complement of the positive real axis to be given by $\exp(s-1)\log(-z)$ with $-\pi < \Im \log(-z) < \pi$ and where the contour C goes just above the positive real axis, circles around 0 in a little circle and comes back just below the positive real axis. Assume this formula. (You need not prove it.) Prove that $\zeta(s)$ can be extended to a meromorphic function on $\mathbb{C}$ whose only pole is a simple pole at $s = 1$.

5 Use either residues or Cauchy's Theorem to compute

$$\int_0^\infty \frac{\sin x}{x}\, dx.$$

6 Let R_n be an arbitrary sequence of nonzero complex numbers such that $|R_n| < 1$ for all n.

a) Write down a convergent infinite series of functions that defines a meromorphic function on $\mathbb{C}$ whose only poles occur at $z = \sqrt{n}$, $n \in \mathbb{N}$, with residues R_n.

b) Prove your series converges away from the poles and defines an analytic function there.