Winter 2015 Geometry / Topology Qualifying Examination

§1. Algebraic Topology

(1) Let $X = S^1 \vee S^1$, choose $x_0 \in X$, and set $\pi_1(X, x_0) = \langle a, b \rangle$. Draw a sketch of the covering $p_H : X_H \to X$ corresponding to the subgroup $H = \langle a, b^2, bab^{-1} \rangle \subset \pi_1(X)$. Is H a normal subgroup or not? How do you see this from the covering space?

(2) Prove that if M is a compact 3-dimensional submanifold of S^3, then $H_1(M; \mathbb{Z})$ is torsion-free.

(3) Prove that a continuous mapping from the 17-dimensional unit ball to itself fixes some point.

(4) (a) Describe a cell decomposition of $\mathbb{R}P^n$ involving one cell of each dimension from 0 to n inclusive.
 (b) Write down the associated cell chain complex of $\mathbb{R}P^5$ with $\mathbb{Z}$ coefficients. Briefly justify your calculation of the boundary maps.
 (c) Calculate $H_*(\mathbb{R}P^5; \mathbb{Z})$.
 (d) Repeat parts (b) and (c) using $\mathbb{Z}/4\mathbb{Z}$ in place of $\mathbb{Z}$.
 (e) Suppose that X is a topological space with the property that $H_*(X; \mathbb{Z}) \approx H_*(\mathbb{R}P^5; \mathbb{Z})$ as graded abelian groups. Determine the cohomology groups of X with $\mathbb{Z}/4\mathbb{Z}$ coefficients. (Do not attempt to describe the multiplicative structure on the cohomology ring. Also note that you do not have a cell decomposition of X, just the isomorphism type of its ordinary homology groups).

§2. Differential Topology

(1) Suppose that X, Y are manifolds and Z is a submanifold of Y. If $f : X \longrightarrow Y$ is a smooth map that is transverse to Z, show that $f^{-1}(Z)$ is a submanifold of X. *You should use the local immersion/submersion theorems. The point of the problem is to construct a suitable locally defined submersion from the transversality condition.*

(2) Suppose that G is a n-dimensional Lie group. A vector field v on G is *left invariant* if

$$(dL_g)_x(v_x) = v_{gx}, \quad \forall x, g \in G,$$

where $L_g : G \longrightarrow G$ denotes left multiplication $x \mapsto gx$. Show that the set $\mathcal{L}$ of all left invariant vector fields on G is an n-dimensional vector space.

(3) Show that the tangent bundle of the torus T^2 is trivial, but the tangent bundle of $T^2 \# T^2$ is not.

(4) Prove that $H^1_{dR}(\mathbb{R}^2 \setminus \{0\}) \neq 0$ from the definition. *Hint: differentiate the angular polar coordinate.*

(5) Without including all the details, briefly(!) outline a proof that if X is a compact oriented manifold, the self-intersection number $I(\Delta, \Delta)$ of the diagonal $\Delta \subset X \times X$ is the Euler characteristic $\chi(X)$.